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Kinetic helicity of vortex in Bose-Einstein condensates

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Abstract. Kinetic helicity of vortex in Bose-Einstein condensates is studied and classified by Hopf index, linking number in geometry. A mechanism of generation and annihilation of vortex line is given by method of phase singularity theory. The dynamic behavior of vortex at the critical points is discussed detailly, and three kinds of length approximation relations at the neighborhood of singularity point are given.

PACS. 03.75.Lm Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices and topological excitations – 67.40.Vs Vortices and turbulence – 03.65.Vf Phases: geometric; dynamic or topological

1 Introduction

Since Lord Kelvin's knotted vortex atom hypothesis [1], knotted vortices have been studied widely in different physical situations such as hydro-dynamics [2,3], field theory [4], nonlinear excited media [5] and optics [6]. Dirac recognized important role of phase singularity of quantum mechanical wave function in his work on monopoles [7], and Madelung gave a vivid interpretation of the lines where the phase is singular [8] on the hydrodynamic formulation of the Schrödinger theory. These are vortex lines in the flow of the probability fluid. In this paper, the knotted vortex lines of Bose-Einstein condensation are studied by method of phase singularity theory.

The dramatic achievement of Bose-Einstein condensation at ultralow temperatures in experiment [9,10] on vapors of rubidium and sodium has stimulated an intense interest in the production of vortices and theoretical investigations of their structure, energy, dynamics and stability [11,12]. The condensates of alkali vapours are pure and dilute, so that the Gross–Pitaevskii (GP) model which represents the so-called mean-field limit of quantum field theories gives a precise description of the atomic condensates and their dynamics at low temperatures.

We known that a single component BEC can be described by a single particle wave function of bosons of mass m. Wave function obeys Gross-Pitaevskii equation [13,14]

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + W_0\psi\left\|\psi\right\|^2. \qquad (1)$$

Here W_0 characterizes the potential between Bosons, assumed positive in our treatment.

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In an external harmonic oscillator trap potential, the energy function can be written as

$$F_{GP} = \int d^2r\left[\frac{\hbar^2}{2m}\left\|\nabla\psi\right\|^2 + \frac{1}{2}\varpi^2r^2\left\|\psi\right\|^2 + \frac{W_0}{2}\left\|\psi\right\|^4\right]. \qquad (2)$$

Here $\frac{1}{2}\varpi^2r^2$ is the confining potential. From (2), one can obtain Gross-Pitaevskii equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + W_0\psi\left\|\psi\right\|^2 + \frac{1}{2}\varpi^2r\psi. \qquad (3)$$

In this paper, we try to give a uniform description of vortex lines in BEC and trapped BEC.

The velocity field $\mathbf{V}$ is defined in terms of the probability current

$$\mathbf{V} = \frac{i\hbar}{2m\psi^*\psi}(\psi\nabla\psi^* - \psi^*\nabla\psi). \qquad (4)$$

The wave function is usually written as

$$\psi = \left\|\psi\right\|e^{i\varphi}, \qquad (5)$$

φ is phase factor. Then velocity is just gradient of phase factor, i.e.

$$\mathbf{V} = \frac{\hbar}{m}\nabla\varphi. \qquad (6)$$

It leads to a trivial curl-free result

$$\nabla\times\mathbf{V} = 0. \qquad (7)$$

Therefore, the flow is strictly irrotational in the bulk. Feynmann found [15] that this statement have to be modified. He point out that curl of velocity can be non-zero at a singular line, the core of quantum vortex line. So vorticity may live only on the lines of singularities of the phase.

Based on the phase singularity theory, we not only obtain the correct result of curl of velocity field, but also get the precise expression of kinetic helicity of vortex lines.

This paper is organized as follows. In the second section, we have classified the topological structure of vortex lines of GP model in terms of Hopf index and Brouwer degree; In the third section, we have studied the topological structure of knotted vortex lines, given the relation of helicity with linking number in geometry. In the fourth section, we have given a mechanism of generation and annihilation of vortex lines.

2 Classification of vortex lines

We denote the condensate wave function as

$$\psi = \psi^1 + i\psi^2. \tag{8}$$

The velocity (4) is written as

$$\mathbf{V} = g\epsilon^{ab}\frac{\psi^a}{\|\psi\|}\nabla\frac{\psi^b}{\|\psi\|}, \qquad a, b = 1, 2. \tag{9}$$

Here ϵ^{ab} is Levi-Civita antisymmetric tensor, and g is $\hbar/m$. The vorticity of the velocity field is $\mathbf{\Omega} = \nabla \times \mathbf{V}$, which can be written in terms of the wave function

$$\Omega^i = \frac{1}{2}g\epsilon^{ijk}\epsilon_{ab}\partial_j\frac{\psi^a}{\|\psi\|}\partial_k\frac{\psi^b}{\|\psi\|}. \tag{10}$$

We use the relation

$$\partial_b\frac{\psi^a}{\|\psi\|} = \frac{\partial_b\psi^a}{\|\psi\|} - \frac{\psi^a\psi^b}{\|\psi\|^3}; \qquad \partial_a\partial_a \ln\|\psi\| = 2\pi\delta^2(\psi), \tag{11}$$

and obtain the vorticity

$$\Omega^i = 2\pi g\delta^2(\psi)D^i\left(\frac{\psi}{x}\right). \tag{12}$$

Here [16,17]

$$D^i\left(\frac{\psi}{x}\right) = \frac{1}{2}\epsilon^{ijk}\epsilon^{ab}\partial_j\psi^a\partial_k\psi^b, \qquad i, j, k = 1, 2, 3; a, b = 1, 2. \tag{13}$$

Equation (13) tells us that the vorticity field

$$\begin{array}{lll} \Omega^i = 0 & \text{only if} & \psi \neq 0, \\ \Omega^i \neq 0 & \text{only if} & \psi = 0. \end{array} \tag{14}$$

Hence,

$$\int_{M_k}\Omega^i d\sigma_i = 2\pi g\beta_k\eta_k. \tag{15}$$

Here M_k is the kth planar element transverse to vortex line L_k with local surface σ_i, and $\eta_k = \mathrm{sgn}\ D(\psi/u) = \pm 1$. It is Brouwer degree of ψ mapping, which characterizes the direction of vortex line. The positive integer number β_k is the Hopf index, which means that when $\mathbf{x}$ covers the zero points region once, the wave function covers the corresponding region in wave function space β_k times. In Moffatt's paper, β_k is also called winding number traced from Gauss. It is obvious that the vortex line can be classified by Brouwer degree and Hopf index, which is also obtained in [18].

In fact, a tensor current can be defined in quantum mechanics, i.e.,

$$T^{\mu\nu} = \frac{1}{2}g\epsilon^{\mu\nu\lambda\varsigma}\epsilon^{ab}\partial_\lambda n^a\partial_\varsigma n^b, \qquad \mu, \nu, \lambda, \varsigma = 0, 1, 2, 3; \quad a, b = 1, 2. \tag{16}$$

Here $1, 2, 3$ denote space coordinate, and 0 denotes time coordinate. The spatial component of tensor current $T^{\mu\nu}$ is

$$T^{0i} = \Omega^i = \frac{1}{2}g\epsilon^{ijk}\epsilon^{ab}\partial_j n^a\partial_k n^b, \qquad i, j, k = 1, 2, 3. \tag{17}$$

It is just vorticity field, i.e., equation (10).

In this section, the topological structure of vortex line is studied, topological structure of vortex lines of Bose-Einstein condensation in terms of Hopf index and Brouwer degree.

3 Kinetic helicity of vortex

The kinetic helicity $\varGamma$ of vortex is $\varGamma = \int \mathbf{V}\cdot\mathbf{\Omega}d^3\mathbf{x}$. From equation (15), we can obtain

$$\varGamma = 2\pi g\sum_{k=1}^{N}\beta_k\eta_k\int_{L_k}V_i dx^i. \tag{18}$$

When these vortex line are closed curves, i.e. a family of knots $\xi_k (k = 1, 2, ...N)$, equation (18) becomes

$$\varGamma = 2\pi g\sum_{k=1}^{N}\beta_k\eta_k\oint_{\zeta_k}V_i dx^i. \tag{19}$$

Linking numbers are the simplest topological relation between two closed curves; this number is zero for two unlinked curves. In order to discuss the linking numbers of the knotted vortex lines, we define Gauss mapping:

$$\tilde{\mathbf{n}}\colon S^1 \times S^1 \to S^2, \tag{20}$$

where $\tilde{\mathbf{n}}$ is a unit vector

$$\tilde{\mathbf{n}}(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{x}_k - \mathbf{x}_l}{\|\mathbf{x}_k - \mathbf{x}_l\|}, \tag{21}$$

where $\mathbf{x}_l$ and $\mathbf{x}_k$ are two points respectively on the knotted vortex lines ξ_l and ξ_k. When $\mathbf{x}_l$ and $\mathbf{x}_k$ are the same point on the same vortex line ζ, $\tilde{\mathbf{n}}$ is just the unit tangent vector. When $\mathbf{x}_l$ and $\mathbf{x}_k$ cover the corresponding vortex lines ξ_j and ξ_k, $\tilde{\mathbf{n}}$ becomes the section of sphere bundle S^2. As in the above section, we can define two two-dimensional unit vectors $\tilde{\mathbf{e}} = \tilde{\mathbf{e}}(\mathbf{x}_l, \mathbf{x}_k)$. $\tilde{\mathbf{e}}, \tilde{\mathbf{n}}$ are normal to each other, i.e.,

$$\begin{aligned} \tilde{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2 &= \tilde{\mathbf{e}}_2 \cdot \tilde{\mathbf{n}} &= \tilde{\mathbf{e}}_2 \cdot \tilde{\mathbf{n}} = 0, \\ \tilde{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1 &= \tilde{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2 &= \tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} = 1. \end{aligned} \tag{22}$$

In fact, the velocity $\mathbf{V}$ can be expressed as

$$V_i = \frac{1}{2} g \epsilon^{ab} e^a \partial_i e^b, \qquad a, b = 1, 2. \tag{23}$$

Substituting it into equation (19), one can obtain new expression of kinetic helicity

$$\Gamma = \pi g^2 \sum_{k=1}^{N} \beta_k \eta_k \oint_{\xi_k} \epsilon^{ab} e^a(\mathbf{x}_l, \mathbf{x}_k) \partial_i e^b(\mathbf{x}_l, \mathbf{x}_k) dx^i. \tag{24}$$

It can be also written as

$$\Gamma = \pi g^2 \sum_{k,l=1}^{N} \beta_k \eta_k \oint_{\xi_k} \oint_{\xi_l} \epsilon^{ab} \partial_i e^a(\mathbf{x}_l, \mathbf{x}_k) \times \partial_j e^b(\mathbf{x}_l, \mathbf{x}_k) dx^i \wedge dx^j. \tag{25}$$

There are three cases: (1) ξ_k, ξ_l are different vortex lines, $\mathbf{x}_l, \mathbf{x}_k$ are different points; (2) ξ_k, ξ_l are the same vortex line, $\mathbf{x}_l, \mathbf{x}_k$ are different points; (3). ξ_k, ξ_l are the same vortex line, $\mathbf{x}_l, \mathbf{x}_k$ are same point. Thus equation (25) can be written as

$$\begin{aligned} \Gamma = 4\pi^2 g^2 \Bigg\{ & \frac{1}{4\pi} \sum_{k=1(\mathbf{x}_l \neq \mathbf{x}_k)}^{N} \beta_k \eta_k \oint_{\xi_k} \oint_{\xi_k} \epsilon^{ab} \partial_i e^a(\mathbf{x}_l, \mathbf{x}_k) \\ & \times \partial_j e^b(\mathbf{x}_l, \mathbf{x}_k) dx^i \wedge dx^j \\ & + \frac{1}{4\pi} \sum_{k=1(\mathbf{x}_l = \mathbf{x}_k)}^{N} \beta_k \eta_k \oint_{\xi_k} \epsilon^{ab} \partial_i e^a(\mathbf{x}_l, \mathbf{x}_k) \\ & \times \partial_j e^b(\mathbf{x}_l, \mathbf{x}_k) dx^i \wedge dx^j \\ & + \frac{1}{4\pi} \sum_{k,l=1(k \neq l)}^{N} \beta_k \eta_k \oint_{\xi_k} \oint_{\xi_l} \epsilon^{ab} \partial_i e^a(\mathbf{x}_l, \mathbf{x}_k) \\ & \times \partial_j e^b(\mathbf{x}_l, \mathbf{x}_k) dx^i \wedge dx^j \Bigg\}. \end{aligned} \tag{26}$$

The first term is just the writhing number [19] $w_r(\xi_k)$ of vortex line ξ_k. The second term is the twisting number $T_w(\xi_k)$ of vortex line ξ_k. From White's formula [20], the self-linking number $S(\xi_k)$ of the vortex line ξ_k is:

$$S(\xi_k) = w_r(\xi_k) + T_w(\xi_k). \tag{27}$$

The third term is Gauss linking number L of vortex lines ξ_k and ξ_l i.e.,

$$L(\xi_k, \xi_l) = \frac{1}{4\pi} \sum_{l=1}^{N} \beta_k \eta_k \oint_{\xi_k} \oint_{\xi_l} \epsilon^{ab} \partial_i e^a(\mathbf{x}_l, \mathbf{x}_k) \times \partial_j e^b(\mathbf{x}_l, \mathbf{x}_k) dx^i \wedge dx^j, \qquad k \neq l. \tag{28}$$

We then obtain the important result

$$\Gamma = 4\pi^2 g^2 \left[\sum_{k=1}^{N} \beta_k \eta_k S(\xi_k) + \sum_{k,l=1}^{N} \beta_k \eta_k L(\xi_k, \xi_l) \right], \tag{29}$$

This result is correct not only in the quantum case [21] but also classical fluid [2]. If there are N filaments with strength χ_k ($k = 1, 2, \cdots N$) whose self knottedness degree, i.e. $\beta_k = 1$ in classical fluid, the kinetic helicity equals $4\pi^2 \sum_{k,l=1}^{N} \eta_k L(\xi_k, \xi_l) = \sum_{k,l=1}^{N} \chi_k \chi_l \eta_k \eta_l \alpha_{kl}$ ($\alpha_{kl} = 1$ if two vortex lines ξ_k, ξ_l are linked; $\alpha_{kl} = 0$, if ξ_k, ξ_l are not singly linked). In the next two sections we will discuss dynamic behavior of vortex lines of of Bose-Einstein condensation which keep the kinetic helicity invariant.

4 Branching of vortex lines

In our work, the evolution of vortex line can be discussed from equation (12). For simplicity, we fix the $x^3 = z$ coordinate and take the XOY plane as the cross-section. The intersection line between vortex line evolution surface and the cross-section is just the motion curve of vortex line. In this two dimensional case, i.e., $\nu = 3$ in equation (16), we obtain

$$\Omega^3 = T^{03} = 2\pi g \delta^2(\psi) D\left(\frac{\psi}{x}\right), \tag{30}$$

$$T^{13} = 2\pi g \delta^2(\psi) D^1\left(\frac{\psi}{x}\right), \tag{31}$$

$$T^{23} = 2\pi g \delta^2(\psi) D^2\left(\frac{\psi}{x}\right). \tag{32}$$

Here

$$\begin{aligned} D\left(\frac{\psi}{x}\right) &= \epsilon^{ab} \partial_1 \psi^a \partial_2 \psi^b, \\ D^1\left(\frac{\psi}{x}\right) &= \epsilon^{ab} \partial_2 \psi^a \partial_t \psi^b, \\ D^2\left(\frac{\psi}{x}\right) &= \epsilon^{ab} \partial_t \psi^a \partial_1 \psi^b. \end{aligned} \tag{33}$$

Please note that $D^1(\psi/x)$ or $D^2(\psi/x)$ in two dimensional case is not the same as one in three dimensional case. Ω^3 is called vortex density, and T^{13}, T^{23} is called vortex density current in Mazenko's paper [17]. It is obvious that the continuity equation is satisfied, i.e.,

$$\partial_t \Omega^3 + \partial_i T^{i3} = 0, \qquad i = 1, 2. \tag{34}$$

The velocity of the intersection point of vortex line and the cross-section is given

$$\frac{dx^i}{dt} = \frac{D^i(\psi/x)}{D(\psi/x)}, \qquad i = 1, 2. \tag{35}$$

In fact, one can find easily that

$$T^{i3} = \Omega^3 \frac{dx^i}{dt}, \qquad i = 1, 2. \tag{36}$$

From equation (35), we know that when $D(\psi/x) = 0$ at very point $(t^*, \mathbf{x}^*)$, the velocity dx^1/dt or dx^2/dt is not unique in the neighborhood of (t^*, x^*). At the singularity point, the normal velocity can not be defined, which is also pointed out by other physicists [6,17]. Because of the conservation of vortex circulation, it should branch or split [22,23]. In reference [18], bifurcation of vortex line is also discussed, but their continuity equation is not correct and time derivative not considered. Taking Taylor expansion of the solution of wave function at the neighborhood of singularity point, one can obtain the direction of zero point on the cross-section at the singularity point. Let us do that now! If we assume that $D^2(\psi/x)_{(t^*,\mathbf{x}^*)} \neq 0$, then there are usually two kinds of singularity points: points where $D^1(\psi/x)\mid_{(t^*,\mathbf{x}^*)} \neq 0$ and points where $D^1(\psi/x)_{(t^*,\mathbf{x}^*)} = 0$.

When $D^1(\psi/x)\mid_{(t^*,\mathbf{x}^*)} \neq 0$, we obtain from equation (35)

$$\frac{dx^1}{dt} = \frac{D^1(\psi/x)}{D(\psi/x)}|_{(t^*,\mathbf{x}^*)} = \infty, \tag{37}$$

i.e.

$$\frac{dt}{dx^1}|_{(t^*,\mathbf{x}^*)} = 0. \tag{38}$$

Taking Taylor expansion of $t = t(x^1, t)$ at this singularity point of vortex line, one can obtain

$$t - t^* = \frac{1}{2}\frac{d^2t}{(dx^1)^2}|_{(t^*,\mathbf{x}^*)}(x^1 - x^{1*})^2 \tag{39}$$

which is a parabola in $x^1 - t$ plane. From equation (39) one can obtain two solutions, which give the branch solutions of vortex line at this critical points. If $d^2t/(dx^1)^2|_{(t^*,\mathbf{x}^*)} > 0$, we have the branch solutions for $t > t^*$, otherwise, we have the branch solutions for $t < t^*$. The former is related to the origin of vortex line at the singularity points. From the continuity equation, we know that the topological number of vortex line is identically conserved. It means that the total topological number of the final vortex lines equals to that of the initial vortex lines. The total numbers of these two generated vortex lines must be zero at the critical point, i.e. the two generated vortex lines have to be opposite, i.e.

$$\beta_1\eta_1 + \beta_2\eta_2 = 0. \tag{40}$$

It is a process of generation or annihilation of vortex lines [24–26]. At the neighborhood of this singularity point, we denote length $l = \Delta x$. From equation (39), one can obtain the approximation relation

$$l \propto \|t - t^*\|^{\frac{1}{2}}. \tag{41}$$

The growth velocity $\gamma = l/\Delta t$ or annihilation velocity of vortex lines

$$\gamma \propto (t - t^*)^{\frac{-1}{2}}. \tag{42}$$

It is obvious that $E_k \propto (t - t^*)^{-1}$ [27]. This result agrees with the numerical data [28,29].

Now let us study the situation of vortex line at its singularity point where $D^1(\psi/x)|_{(t^*,\mathbf{x}^*)} = 0$. The Taylor expansion of the solution of $\psi^1$and ψ^2 in the neighborhood of this singularity point can generally be denoted as $A(x^1 - x^{1*})^2 + 2B(x^1 - x^{1*})(t - t^*) + C(t - t^*)^2 + \cdots = 0$, where A, B and C are three constants. Then from the Taylor expansion, we can obtain

$$A\left(\frac{dx^1}{dt}\right)^2 + 2B\frac{dx^1}{dt} + C = 0. \tag{43}$$

There are two kinds of length scales:

Case 1, $A \neq 0, (B^2 - AC) \geqslant 0$. At the neighborhood of the singularity point, we denote scale length $\Delta x = l$. Then from the Taylor expansion, we can obtain the asymptotic relation

$$l \propto (t - t^*). \tag{44}$$

Case 2, $A = C = 0$. We obtain

$$\frac{dt}{dx^1}\bigg|_{(t^*,\mathbf{x}^*)} = 0, \quad \text{or} \quad \frac{dx^1}{dt}\bigg|_{(t^*,\mathbf{x}^*)} = 0. \tag{45}$$

From equation (45), one can obtain

$$l = const, \qquad \gamma = 0. \tag{46}$$

It is obvious that vortex lines are relatively rest when $l = const$.

5 Conclusion

We denote total topological number Q of vortex lines configuration

$$Q = \sum_{k=1}^{N}\beta_k\eta_k S(\xi_k) + \sum_{k,l=1}^{N}\beta_k\eta_k L(\xi_k, \xi_l), \tag{47}$$

which is a Hopf invariant, and also called topological charge by Faddeev. Then

$$\Gamma = 4\pi^2 Q. \tag{48}$$

Since the kinetic helicity Γ is invariant in our case, then the sum of the the final vortex topological number must be equal to that of the original vortex lines at the singularity point, i.e.

$$Q = cons\tan t. \tag{49}$$

This relation and the singularity condition determine the dynamic behavior of the vortex lines. The situation becomes complicated for the the entangleness of vortex lines.

The branching condition of vortex line is determined by wave function $\psi = 0$, and

$$D\left(\frac{\psi}{x}\right) = 0, \tag{50}$$

or

$$D^1\left(\frac{\psi}{x}\right) = 0. \tag{51}$$

We know that wave function satisfies GP equation, i.e., equation (1) in BEC, or equation (3) in trapped BEC. The branching of vortex in BEC is different with that in trapped BEC naturally because of the confining potential.

In the present work, kinetic helicity of vortex lines of GP model are classified by Hopf index, Brouwer degree and linking number in geometry. A mechanism of generation and annihilation of vortex line is given. The evolution equation of vortex line has been given and its dynamic behavior at the singularity points is discussed in detail. We result that there are only three kind of length approximation relation at the neighborhood of singularity point in this model, i.e. $l \propto (t-t^*)^{1/2}$, $l \propto t-t^*$, $l = const$. The dynamic behavior becomes complicated because of the the entangleness of vortex lines. The entanglement of vortex lines may be verified by experiment in future.

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